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A Set of Assumptions for Projective Geometry.[*]

BY OSWALD VEBLEN AND JOHN WESLEY YOUNG.

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Introduction.

This paper contains a completely independent[†] set of assumptions for projective geometry stated in terms of undefined elements called points and undefined classes of points called lines. The assumptions are so arranged that a certain group of eight characterize what may be called *general projective spaces, i. e.*, spaces in which the points can be represented by homogeneous coordinates which are elements of a finite or infinite number-system, in which the operation of multiplication may or may not be commutative. On adding to this group an assumption (like our Assumption P, p. 352) from which can be proved the fundamental theorem of projectivity (in the form given, for example, on p. 352), we obtain a set of assumptions which characterize the most general projective spaces properly so-called, *i. e.*, spaces in which the points may be represented by homogeneous coordinates which are elements of a commutative number-system, *i. e.*, of a finite or infinite field.

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[*] Presented to the American Mathematical Society, Dec. 27, 1907.

[†] Ordinally independent sets have been given before, but so far as the authors are aware this is the first completely independent set.

Modular and non-modular spaces, *i. e.*, spaces in which the coordinates are elements of modular or non-modular fields, are distinguished by means of Assumption H (§ 4). Finally it is shown how by replacing Assumption P by assumptions of continuity and closure we may arrive at categorical and completely independent sets of assumptions on the one hand for the projective space in which the coordinates are ordinary real, and on the other hand for that in which the coordinates are ordinary complex numbers.

A complete list of the assumptions for the ordinary real and complex projective spaces of three dimensions will be found at the beginning of § 9.

The obligations of the authors to previous work will be evident to any one who is familiar with the literature of the subject. For this reason we have omitted detailed references to previous work and content ourselves with the reference to the article of ENRIQUES, Prinzipien der Geometrie, in the *Encyklopaedie der Mathematischen Wissenschaften*, Band III, Part I, pp. 1–129, for a bibliography. For a similar reason we have omitted all proofs of the early theorems, believing that their derivation from the assumptions in question is sufficiently familiar. The definitions of many well-established terms have likewise been omitted in the interest of brevity.

§ 1. *The Assumptions for General Projective Geometry.*

In the following assumptions for projective geometry we have chosen the *point* and the *line* as undefined elements, the line being regarded as an undefined *class of points*. The only undefined relation used is that of *belonging to a class*. This relation will be variously expressed by such phrases as: a point is on a line; a line joins two points; three points are collinear; etc. In this section we give a set of assumptions that define what may be called *general projective spaces*, in which the points may be represented by homogeneous coordinates which are elements of a finite or infinite number-system, in which multiplication may or may not be commutative.

THE ASSUMPTIONS OF ALIGNMENT, A:

A1. *If A and B are distinct points there is at least one line containing both A and B.*

A2. *If A and B are distinct points there is not more than one line containing both A and B.*

A3. *If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.* (Fig. 1.)

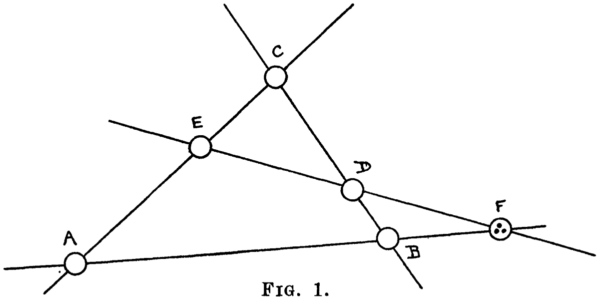

FIG. 1.

AN ASSUMPTION OF EXTENSION, E:

E0. *There are at least three points on every line.*[*]

From A1 and A2 follows readily:

THEOREM 1. *Two distinct points determine one and only one line. If C and D are distinct points of the line AB,*[†] *then A and B are points of the line CD. Two distinct lines can not have more than one point in common.*

Definition. If P, Q, R are three points not on the same line, and l is a line joining Q and R, the class S_2 of all points such that every point of S_2 is collinear with P and some point of l is called the *plane* determined by P and l. If P, Q, R, T are four points not in the same line or plane, and if α is a plane containing Q, R and T, the class S_3 of all points such that every point of S_3 is collinear with P and some point of α is called the *three-space* determined by P and α.

It is now possible to derive readily from the set of assumptions given above the results contained in the following theorems:

THEOREM 2. *If A and B are points of a plane, every point of the line AB is a point of the plane. Any two lines lying in the same plane have a point in common.*

---

[*] This excludes merely the case of a space in which every line consists of only two points.

[†] The symbol AB implies $A \neq B$ and denotes the line determined by A and B.

The plane α determined by a point P and a line l is identical with the plane determined by a point Q and a line m, if Q and m are on α. There is one and only one plane containing three given non-collinear points.

THEOREM 3. *If A and B are distinct points of a three-space, every point of the line AB is a point of the three-space. If a plane α and a line l not on α lie wholly in the same three-space, then α and l have one and only one point in common. Any two distinct planes of a three-space have one and only one line in common.*

These three theorems are meaningless unless there exists at least one line (Theorem 1), or one plane (Theorem 2), or one three-space (Theorem 3). We could proceed to define a four-space, five-space, ..., n-space in a manner analogous to the definitions of a two-space (plane) and three-space already given. The fundamental properties of alignment of such spaces can be derived without difficulty from the assumptions stated. A set of assumptions, however, from which the properties of a space of given dimensionality are to be derived, should contain in addition to those already stated such assumptions of extension and closure as will insure the existence of the space in question and exclude spaces of higher dimensionality. In this paper we confine ourselves to three dimensions. There follow accordingly for this case the necessary

ASSUMPTIONS OF EXTENSION AND CLOSURE,[*] E:

E1. *There exists at least one line.*

E2. *It is not true that every point lies on every line.*

E3. *It is not true that every point lies on every plane.*

E3′. *If S is a three-space, every point lies in S.*

It is now a simple matter to derive the principle of duality in a three-space and in a plane, in view of the fact that the duals of the assumptions can be proved without difficulty. These two principles are stated in the following two theorems:

THEOREM 4: THE PRINCIPLE OF DUALITY FOR A THREE-SPACE. *Any proposition deducible from assumptions* A *and* E *is valid if the words "point" and "plane" are interchanged.*

THEOREM 5: THE PRINCIPLE OF DUALITY IN A PLANE. *Any proposition concerning points and lines of the same plane derived from assumptions* A *and* E, *is valid if the words point and line are interchanged.*

---

[*] The words "extension" and "closure" in this connection were suggested by N. J. LENNES.

This brief statement of the principle of duality makes necessary the use of such expressions as "a line lying on a point," "a plane lying on a point or a line," "a point containing a plane" etc., in a sense that need not be further explained here.

It is now possible to enumerate the fundamental geometric forms, and to define perspectivity and projectivity* in the usual manner. In what follows we omit most definitions of well-established terms. Such terms as are defined, moreover, and the theorems that are proved will be confined in general to one form; the dual definitions and theorems are everywhere implied without being explicitly stated.

Of the theorems derivable from those thus far noted we mention, first:

THEOREM 6: THE THEOREM OF DESARGUES. *The intersections of the pairs of homologous sides of two perspective triangles are collinear.*

Definition. The set of points in which the sides of a complete quadrangle meet a line is called a *quadrangvlar set;* it is denoted by the symbol $Q(A, B, C; D, E, F)$, which implies that A, D; B, E; C, F are the intersections of pairs of opposite sides of the quadrangle with AB and that A, B, C are the intersections with AB of three concurrent sides of the quadrangle. In case $B = E$ and $C = F$, A and D are *harmonic conjugates* with respect to B and C.

From Theorem 6 then follows:

THEOREM 7. *If all but one of the points of a quadrangular set* $Q(A, B, C; D, E, F,)$ *are given, the remaining one is uniquely determined. In particular, the harmonic conjugate of a point with respect to two others is uniquely determined.*

The following propositions concerning the projectivities of one-dimensional forms are also readily derivable from the assumptions and theorems thus far noted:

THEOREM 8. *If* A, B, C, D *are points of a line, and* A', B', C' *are points of another or the same line, we always have* $(A, B, C) \barwedge (A', B', C')$† *and* $(A, B, C, D) \barwedge (B, A, D, C)$. *A set of collinear points which is projective with a quadrangular set is a quadrangular set. In particular, if one of two projective sets*

---

* We use PONCELET'S definition of projectivity, which defines it as the resultant of a sequence of perspectivities.

† The notation $(A, B, \ldots.) \barwedge (A', B', \ldots.)$ denotes a projectivity in which A, A'; B, B'; .... are homologous pairs. Similarly $(A, B, \ldots.) \overset{P}{\doublebarwedge} (A', B', \ldots.)$ denotes a perspectivity with center P in which A, A'; B, B'; .... are homologous pairs.

of four collinear points is harmonic, so also is the other. If the ranges on two pairs of a set of three concurrent lines are perspective, so also are the ranges on the third pair.

It is not possible, however, to deduce from the assumptions A and E the so-called fundamental theorem of projectivity, which we state in the following form:

THE FUNDAMENTAL THEOREM OF PROJECTIVITY. *If A, B, C, D are distinct points of a line, and A', B', C' any three distinct points of another or the same line, then for any projectivities giving $(A, B, C, D) \barwedge (A', B', C', D')$ and $(A, B, C, D) \barwedge (A', B', C', D'_1)$ we have $D' = D'_1$.*

To derive the fundamental theorem another assumption is necessary, which may take any one of several forms. One form is the following:

AN ASSUMPTION OF PROJECTIVITY, P:

P. *Two projective ranges of points on two different lines which have a self-corresponding point are perspective.*

Very little use of this assumption is made in the subsequent parts of this paper; indeed *the principal part of the paper is entirely independent of it, so that all numbered theorems are derivable without its use.* We have given it here merely in order that we might characterize by a set of simple assumptions what may be called the most general *properly projective spaces; i. e.,* those in which the fundamental theorem of projectivity is valid. Such a space is characterized by assumptions A, E and P. A space satisfying assumptions A and E, and not P, may then be called an *improperly projective space.* Cf., in this connection, Theorem 14 below, which shows that assumption P is equivalent to the commutative law of multiplication in the algebra there developed.

§ 2. *Algebra of Points and the Introduction of Analytic Methods.*

At this point it seems desirable to introduce analytic methods. The introduction of a point algebra, which is possible without the use of any further assumptions, will throw more light on the preceding results and will greatly facilitate much of the subsequent work.

Given a line l and on l three distinct (arbitrary) fixed points which for convenience and suggestiveness we denote by P_0, P_1, P_∞, we define two *one-*

*valued operations** on pairs of points of l with reference to the *fundamental points* P_0, P_1, P_∞. The fundamental points are said to determine the *scale* (P_0, P_1, P_∞) on l.

Definition. The point P_{x+y} determined by the relation $Q(P_\infty, P_x, P_0; P_\infty, P_y, P_{x+y})$ is called the *sum* of the two points P_x and P_y (in symbols $P_x + P_y = P_{x+y}$) in the scale (P_0, P_1, P_∞) on l. (Cf. Fig. 2.) The point P_{xy} determined by the relation $Q(P_0, P_1, P_x; P_\infty, P_{xy}, P_y)$ is called the *product* of P_x by P_y (in symbols $P_x \cdot P_y = P_{xy}$) in the scale (P_0, P_1, P_∞) on l. (Cf. Fig. 3.)

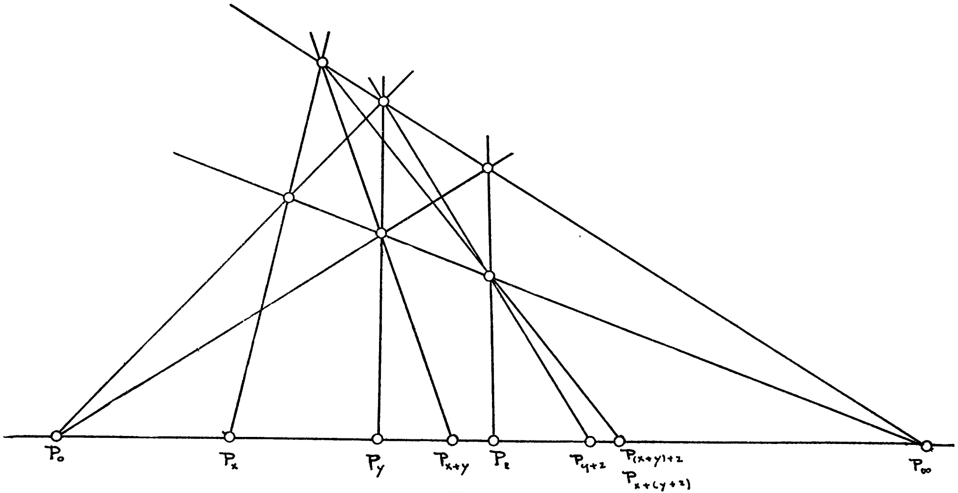

FIG. 2.

From Theorem 7 follows:

THEOREM 9. *The operations of addition and multiplication are one-valued, except for* $P_0 \cdot P_\infty$ *and* $P_\infty \cdot P_0$.

From Theorem 7 likewise follows:

THEOREM 10. *The operation of addition is commutative.*

There is no difficulty, moreover, in proving

THEOREM 11. *The operations of addition and multiplication are associative.*

For, the constructions for $(P_x + P_y) + P_z$ and $P_x + (P_y + P_z)$ can easily be so made that they are both defined by the intersection of the same line with l. Similarly for $P_x \cdot (P_y \cdot P_z)$ and $(P_x \cdot P_y) \cdot P_z$. (Cf. Figs. 2, 3.)

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*By a *one-valued* operation o on a pair of points A, B is meant any process whereby with every pair A, B is associated a point C, which is unique provided the order of A, B is given; in symbols, $AoB = C$. Here "order" has no geometrical significance, but implies merely the formal difference of AoB and BoA. If $AoB = BoA$ the operation is *commutative*; if $(AoB)oC = Ao(BoC)$, *associative*.

By means of assumptions A and E alone we may also derive the following important theorem:

THEOREM 12. *Between the points P_x, P_y, P_{xy} we always have the projectivities*

$$(P_\infty, P_0, P_1, P_x) \overline{\wedge} (P_\infty, P_0, P_y, P_{xy})$$

and

$$(P_\infty, P_0, P_1, P_y) \overline{\wedge} (P_\infty, P_0, P_x, P_{xy}).$$

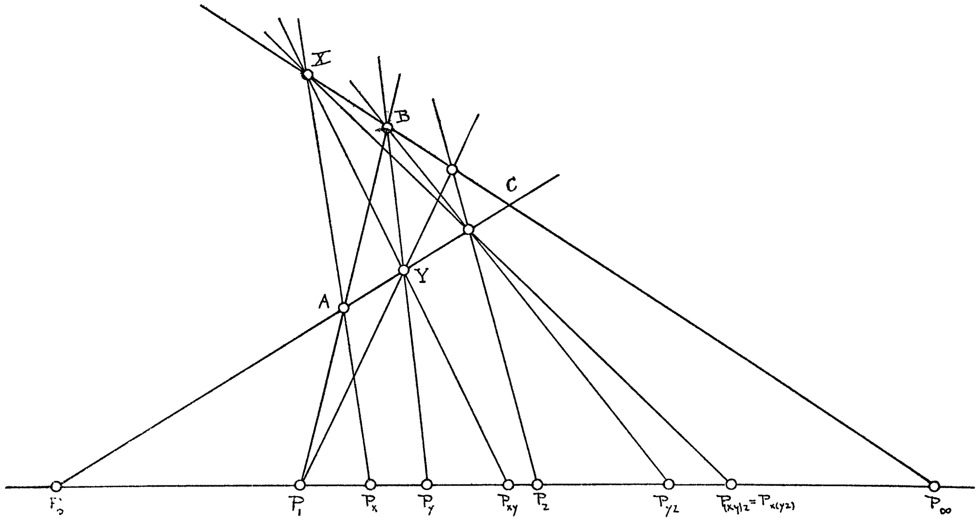

FIG. 3.

Proof. Let the quadrangle $ABXY$ define the point P_{xy}. (Fig. 3.) We then have

$$(P_\infty, P_0, P_1, P_x) \overset{A}{\underset{\wedge}{=}} (P_\infty, C, B, X) \overset{Y}{\underset{\wedge}{=}} (P_\infty, P_0, P_y, P_{xy});$$

and also

$$(P_\infty, P_0, P_1, P_y) \overset{B}{\underset{\wedge}{=}} (C, P_0, A, Y) \overset{X}{\underset{\wedge}{=}} (P_\infty, P_0, P_x, P_{xy}).$$

From this theorem we can readily derive

THEOREM 13: THE DISTRIBUTIVE LAW. *For any three points P_x, P_y, P_z on l we have $P_x \cdot (P_y + P_z) = P_x \cdot P_y + P_x \cdot P_z$ and $(P_y + P_z) \cdot P_x = P_y \cdot P_x + P_z \cdot P_x$.*

Proof. By Theorem 12 we have

$$(P_\infty, P_0, P_1, P_y, P_z, P_{y+z}) \overline{\wedge} (P_\infty, P_0, P_x, P_{xy}, P_{xz}, P_{x(y+z)});$$

also $Q(P_\infty, P_y, P_0;\ P_\infty, P_z, P_{y+z})$; whence we have $Q(P_\infty, P_{xy}, P_0;\ P_\infty, P_{xz}, P_{x(y+z)})$ (Theorem 8), which gives $P_{xy} + P_{xz} = P_{x(y+z)}$. This is the first relation of the theorem. The second is obtained similarly.

The commutative law of multiplication can not be derived from assumptions A and E alone. The intimate connection between the commutative law of multiplication and the fundamental theorem of projective geometry is expressed in the following:

THEOREM 14. *Multiplication is commutative, if and only if the fundamental theorem of projective geometry is valid.*

Proof. From Theorem 12 we have

$$(P_\infty, P_0, P_1, P_x) \barwedge (P_\infty, P_0, P_y, P_{xy})$$

and

$$(P_\infty, P_0, P_1, P_x) \barwedge (P_\infty, P_0, P_y, P_{yx}),$$

whence clearly $P_{xy} = P_{yx}$, if and only if the fundamental theorem holds. (Cf. p. 352.)

The inverse operations, subtraction and division, may now be defined in the usual manner. It is then readily seen that *the points of a line on which a scale has been established form a number-system,*[*] *if the point P_∞ be excluded, in which the points P_0 and P_1 play the rôle of zero and unity respectively.* For the definitions of addition and multiplication give at once

$$P_0 + P_x = P_x + P_0 = P_x, \quad P_0 \cdot P_x = P_x \cdot P_0 = P_0,$$

and

$$P_1 \cdot P_x = P_x \cdot P_1 = P_x, \text{ if } P_x \neq P_\infty.$$

This number-system is commutative, if and only if the space considered is properly projective. For convenience we shall denote the points of a line by the small letters of the alphabet, whenever we think of them as numbers of a number-system.

We may now treat analytically the projectivities on a line *for the case in which the number-system is commutative, i. e.,* for a properly projective space. It is readily seen from the definitions that each of the transformations

$$x' = x + a, \quad x' = ax, \quad x' = 1/x \tag{1}$$

defines a projectivity; and it is readily shown that every transformation of the form

$$x' = \frac{ax + b}{cx + d} \qquad (ad - bc \neq 0) \tag{2}$$

can be resolved into the product of transformations of types (1), so that every transformation (2) is a projectivity. That every projectivity in a properly projective space can be represented by (2) then follows at once from the fact that any such projectivity is uniquely defined when three pairs of homologous points are given. This leads to three linear homogeneous equations for the determination of the ratios $a:b:c:d$, and these equations are necessarily solvable in the field.

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[*] For the general definition of a number-system see DICKSON, Definition of Linear Associative Algebra by Independent Postulates, *Trans. Amer. Math. Soc.*, Vol. IV (1903), p. 21.

The double ratios of four points are now defined in the usual manner and their invariance under projective transformations follows immediately from their evident invariance under each of the three types (1). Further, the double ratio of a harmonic form (a, b, c, d) in which a, c are conjugate is clearly

$$\frac{a-b}{c-b} : \frac{a-d}{c-d} = -1, \tag{3}$$

since -1 is the harmonic conjugate of 1 with respect to 0 and ∞ (by definition of -1 as $0-1$) and all harmonic forms are projective.

The exceptional character of the point P_∞ in the point-algebra may be removed in the usual manner by the introduction of homogeneous coordinates and the ordinary analytic methods may be developed for the plane and for space without difficulty.

§ 3. *Nets of Rationality.*

Definition. A point P of a line is said to be *harmonically* (*quadrangularly*) *related to three given distinct points* A, B, C of the line, provided P is one of a sequence of points $A, B, C, H_1, H_2, H_3, \ldots$ of the line, finite in number, such that H_1 is the harmonic conjugate of one of the points A, B, C with respect to the other two, and such that every other point H_i is harmonic with three of (is one of a quadrangular set of which the other five belong to) the set $A, B, C, H_1, H_2, \ldots, H_{i-1}$. The class of all points harmonically related to three distinct points A, B, C on a line is called the *net of rationality* (on the line) defined by A, B, C; it is denoted by $R(A, B, C)$.

THEOREM 15. *If A, B, C, D and A', B', C', D' are respectively points of two lines such that $(A, B, C, D) \barwedge (A', B', C', D')$, and if D is harmonically (quadrangularly) related to A, B, C, then D' is harmonically (quadrangularly) related to A', B', C'.*

This follows directly from the fact that the projectivity of the theorem makes the set of points H_j which defines D as harmonically (quadrangularly) related to A, B, C projective with a set of points H_i' such that every harmonic (quadrangular) set of points of the sequence $A, B, C, H_1, H_2, \ldots, D$ is homologous with a harmonic (quadrangular) set of the sequence $A', B', C', H_1', H_2', \ldots, D'$ (Theorem 8).

COROLLARY. *If a class of points on a line is projective with a net of rationality on a line, it is itself a net of rationality.*

THEOREM 16. *If K, L, M are three distinct points of $R(A, B, C)$, A, B, C are points of $R(K, L, M)$.*

Proof. From the projectivity $(A, B, C, K) \overline{\wedge} (B, A, K, C)$ follows by Theorem 15 that C is a point of $R(A, B, K)$, or $R(A, B, C) = R(A, B, K) = R(A, K, M) = R(K, L, M)$.

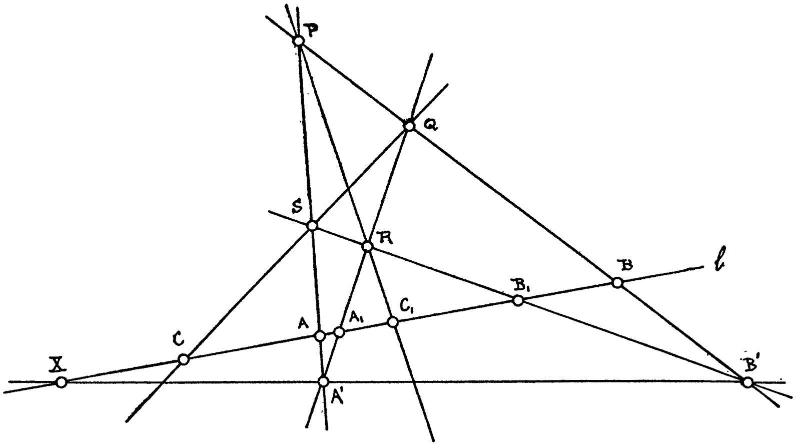

FIG. 4.

COROLLARY. *A net of rationality on a line is determined by any distinct three of its points.*

THEOREM 17. *If all but one of the six (or five, or four) points of a quadrangular set are points of the same net of rationality R, this one point is also a point of R.*

Proof. Let the sides of the quadrangle $PQRS$ (Fig. 4) meet the line l as indicated in the points A, A_1; B, B_1; C, C_1; and suppose that the first five of these are points of a net of rationality $R = R(A, A_1, B_1) = R(A_1, B_1, C) = \ldots$. We must prove that C_1 is a point of R. Let the pairs of lines PS, QR and PQ, RS meet in A', B' respectively, and let $A'B'$ meet l in X. We then have

$$(A_1, B, B_1, C) \overset{Q}{\overline{\overline{\wedge}}} (R, B', B_1, S) \overset{A'}{\overline{\overline{\wedge}}} (A_1, X, B_1, A),$$

whence $(A_1, B, B_1, C) \barwedge (A_1, X, B_1, A)$, so that if B is a point of $R(A_1, B_1, C)$, X is a point of $R(A_1, B_1, A)$; but these two nets are identical with R, so that X is a point of R. Now,

$$(A, B_1, X, A_1) \overset{A'}{\doublebarwedge} (S, B_1, B', R) \overset{P}{\doublebarwedge} (A, B_1, B, C_1),$$

which shows that C_1 is a point of R.

COROLLARY. *The class of all points quadrangularly related to three distinct points A, B, C is $R(A, B, C)$.*

Although the fundamental theorem of projective geometry can not be deduced in general from the assumptions A and E, the corresponding theorem for a net of rationality on a line follows almost immediately from the preceding theorems. It may be stated as follows:

THEOREM 18: THE FUNDAMENTAL THEOREM OF PROJECTIVITY FOR A NET OF RATIONALITY ON A LINE. *If A, B, C, D are distinct points of a net of rationality R on a line, and A', B', C', any three distinct points on another or the same line, then for any projectivities giving $(A, B, C, D) \barwedge (A', B', C', D')$ and $(A, B, C, D) \barwedge (A', B', C', D_1')$ we have $D' = D_1'$.*

Proof. Let Π and Π_1 be the two projectivities respectively. Then clearly the projectivity $\Pi_1 \Pi^{-1}$ leaves A', B', C' unchanged and transforms D' into D_1'. But it is easy to see that a projectivity which leaves three distinct points of a line unchanged leaves all the points of the net of rationality defined by these points unchanged, since if three points of a line are fixed the harmonic conjugate of one with respect to the other two is also fixed.

COROLLARY. *If two nets of rationality on different lines are projective and have a self-corresponding point, they are perspective.*

Definition. If A, B, C, D are the vertices of a quadrangle, a point P of their plane is said to be *rationally related* to them, if P is one of a sequence of points $A, B, C, D, D_1, D_2, \ldots$ finite in number, such that D_1 is a diagonal point of the original quadrangle and such that every other point D_i is a diagonal point of a quadrangle whose vertices are contained in the set

$$A, B, C, D, D_1, D_2, \ldots, D_{i-1}.$$

A line is said to be rationally related to A, B, C, D, if it joins two points rationally related to them. The class of all points and lines rationally related to four distinct coplanar points is called the *net of rationality* (in the plane) defined by the four points. It is denoted by $R(A, B, C, D)$.

The following is a consequence of this definition and the corollary of Theorem 17:

THEOREM 19. *The points in which the lines of a net of rationality in a plane meet any line of the plane form a net of rationality on this line.*

Definition. If A, B, C, D, E are the vertices of a complete space five-point, a point P is said to be *rationally related* to them, if P is one of a sequence of points $A, B, C, D, E, I_1, I_2, I_3, \ldots$, finite in number, such that I_1 is the intersection of three distinct faces of $ABCDE$, and such that every other point I_i is the intersection of three distinct faces of a complete space five-point whose vertices belong to the set $A, B, C, D, E, I_1, I_2, \ldots, I_{i-1}$. A line is said to be rationally related to A, B, C, D, E if it joins two, a plane if it joins three non-collinear, points which are rationally related to A, B, C, D, E. The set of all points, lines, and planes rationally related to A, B, C, D, E is called the *net of rationality* (in space) defined by A, B, C, D, E; it is denoted by $R(A, B, C, D, E)$.

This definition gives

THEOREM 20. *The points and lines (points) in which the lines and planes (planes) of a net of rationality in space meet any plane (line) form a net of rationality on this plane (line).*

Theorems analogous to Theorems 15, 16, 18 can readily be derived for nets of rationality in a plane and in space.

This leads to the important result:

THEOREM 21. *A net of rationality in space is a space satisfying the assumptions* A *and* E *and also* P; *i. e., a net of rationality in space is a properly projective space.*

COROLLARY. *If P_0, P_1, P_∞ are three distinct points of any line, the points of $R(P_0, P_1, P_\infty)$ form a commutative number-system or field.*

This follows directly from the last theorem in connection with Theorem 14.

"Rational" geometries would result, if we added to our assumptions A and E another assumption of closure (E3′(r)) to the effect that all the points of space belong to the same net of rationality. In general, any five-point in any properly or improperly projective space determines a sub-space which is rational and therefore properly projective.

§ 4. *Assumption* H *and the Definition of Separation.*

Definition. Any sequence of points $\ldots, H_0, H_1, H_2, H_3, \ldots$ on a line is called a *harmonic sequence,* if it has the properties: 1) that the middle one of any three consecutive points of the sequence is the harmonic conjugate, with respect to the other two, of a fixed point H of the line; and 2) that, if H_i, H_{i+1} are any two consecutive points of the sequence, the harmonic conjugate of H_i with respect to H_{i+1} and H is a point of the sequence. The point H is called the *limit point* of the sequence. (Cf. Fig. 5.)

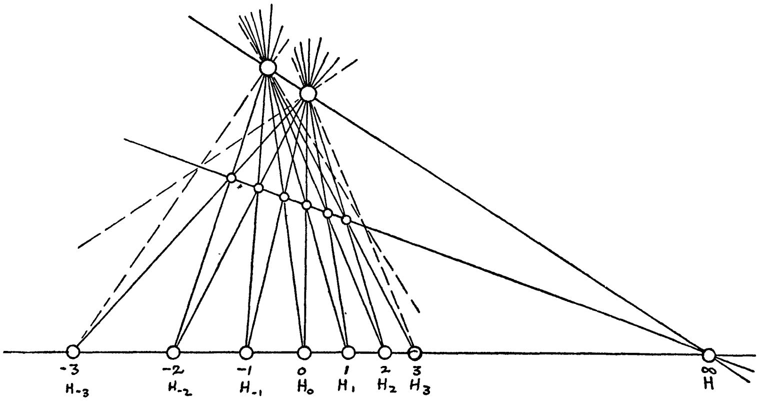

FIG. 5.

If the limit point of a harmonic sequence is associated with ∞* and two successive points of the sequence with 0 and 1 respectively, it follows (cf. Fig. 5) at once, from the definitions of § 2, that the sequence consists of the points

$$\ldots\ldots, -1-1-1, -1-1, -1, 0, 1, 1+1, 1+1+1, \ldots\ldots;$$

or if we adopt the usual symbols to denote these numbers, of the points

$$\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots$$

It should here be noted that the assumptions thus far made do not imply that this sequence contains an infinite number of points.

Clearly all points of a harmonic sequence belong to the same net of rationality. Moreover, it follows from Theorem 21, corollary, that if x and y belong to the net $R(0, 1, \infty)$ so also do $x+y$, $x-y$, xy, and x/y, so that $R(0, 1, \infty)$ contains all numbers that can be obtained from 0, 1 by a finite number of the rational operations. Further, from (3)† (p. 356) of § 2, it follows that the fourth harmonic of any point in $R(0, 1, \infty)$ with respect to two others can be obtained by a finite number of rational operations on a, b, c. Whence follows that *the number-system associated with every net of rationality consists of all numbers that can be obtained by a finite number of rational operations on* 0 *and* 1, *and only these.*

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* For convenience we use the symbols $0, 1, \infty, \ldots\ldots, x, y, \ldots\ldots$ in place of $P_0, P_1, P_\infty, \ldots\ldots, P_x, P_y, \ldots\ldots$

† (3) is clearly applicable, since multiplication is commutative in any net of rationality.

Returning to the harmonic sequence, two possibilities present themselves: Either all the points of a harmonic sequence are distinct from their predecessors, in which case the number-system contains all the ordinary rational numbers; or some point of the sequence coincides with one of the preceding points, in which case the number-system consists of all integers mod. p (p being any prime).* The net of rationality may in this case be called *modular.* These results we combine as follows:

THEOREM 22. *Every net of rationality determines a number-system which consists either of all integers mod.* p *(*p *any prime), or of the set of all rational numbers. In the first case the whole space in which the net lies may contain either a finite or an infinite number of points, but it has the same modulus for all of its nets of rationality. In the second case the whole space and all of its nets of rationality are infinite.*

COROLLARY. *Any (not necessarily commutative) number-system is such that any two numbers* a, b, *(e. g.,* 0, 1*) determine a set of numbers rationally related to them which is either finite and prime or infinite and isomorphic with the set of all rationals.*

Working now toward the characterization of the ordinary real and complex projective spaces, we eliminate the possibility of a modular number-system by the following:

ASSUMPTION H:

H. *If there is any harmonic sequence, there is one such that every point of it is distinct from all the points of the sequence that precede it.*†

By virtue of this assumption we have clearly:

THEOREM 23. *The points of any net of rationality on a line give rise to a number-system which is simply isomorphic with the field of all rational numbers.*

We proceed to define a fundamental relation between pairs of points of a net of rationality on a line for which H is satisfied:

Definition. Two points A, C of a non-modular net of rationality on a line are said to *separate* two others B, D of the net (in symbols $AC \parallel BD$), if and only if the assignment of the numbers 0, 1, ∞ to the points A, B, C respectively assigns a negative number to D.

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* The modulus must be a prime number, since division must be always possible.

† This has as part of its content "Fano's Axiom," that the diagonal points of a complete quadrangle are non-collinear. Cf. GINO FANO, *Gior. di Mat.*, Vol. XXX (1892), p. 106.

This definition is dependent on the order in which the points are taken. The following theorem shows, however, that the relation of separation is independent of the order of the pairs of points or of the order of the points within the pair:

THEOREM 24. 1) *The relation* $AC \parallel BD$ *implies the relations* $BD \parallel AC$ *and* $AC \parallel DB$, *and excludes the relation* $AB \parallel CD$. 2) *Given any four distinct points of a net of rationality on a line, we have either* $AB \parallel CD$, *or* $AC \parallel BD$, *or* $AD \parallel BC$. 3) *From the relations* $AC \parallel BD$ *and* $AD \parallel CE$ *follows the relation* $AD \parallel BE$.*

This theorem follows at once from the following two:

THEOREM 25. *If* $AC \parallel BD$ *and* $(A, B, C, D) \barwedge (A', B', C', D')$, *then also* $A'C' \parallel B'D'$.

Proof. Since any projectivity transforms every quadrangular set into a quadrangular set, it is clear that the number assigned to D' by the assignment of 0, 1, ∞ to A', B', C' must be precisely the same as the number assigned to D by the assignment of 0, 1, ∞ to A, B, C.

THEOREM 26. *Two points* a, c *of the net* $R(0, 1, \infty)$ *separate two others* b, d *of this net if and only if one and only one of the numbers* a, c *lies between the two numbers* b, d.

Proof. If we project a, b, c, supposed finite, into 0, 1, ∞ respectively by the transformation

$$x' = \frac{b-c}{b-a} \cdot \frac{x-a}{x-c},$$

it is readily seen that x' is negative if and only if one of the numbers a, c, and only one, lies between b and x. The necessary modification of this argument in case one of the numbers a, b, c, d is ∞ is obvious.

COROLLARY. *Two harmonic pairs always separate each other.*

§ 5. *The Assumption of Continuity and the Definition of a Chain.*

Definition. Given three distinct points A, B, C of a net of rationality on a line, the *segment* ABC (seg ABC) of the net consists of B and all points X of

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* The properties expressed in this theorem are sufficient to define abstractly the relation of separation. Cf. VAILATI, *Révue de Mathématiques*, Vol. V, pp. 76, 183; also, PADOA, *Révue de Mathématiques*, Vol. V, p. 185.

the net such that A, C do not separate B, X. The totality of points Y such that A, C do separate B, Y constitutes the *segment complementary to* seg ABC. The points A, C are called the extremities of each of the two segments.

Clearly seg ABC and seg CBA contain the same points.

Any two distinct points of a net of rationality on a line divide the net into two segments S and S' such that the two given points separate every pair of points of which one belongs to S and the other to S', and such that no pair of points of S separates any pair of points of S'. It is clear also that any point P of a segment S (of a net of rationality on a line) of which A and C are extremities divides the segment S into two segments S_1, S_2 such that no pair of points of S_1 separates any pair of points of S_2, and such that the pair AP and the pair PC each separates every pair of points of S, of which one belongs to S_1 and the other to S_2.

Definition. Any division of the points of a non-modular net of rationality on a line into two classes K_1 and K_2 such that

1) Every point of the net belongs either to K_1 or to K_2,

2) No pair of points of K_1 separates any pair of K_2,

is called a *cut* in the net. The classes K_1, K_2 are called the *sides* of the cut.

Any two distinct points of a net of rationality on a line determine a cut, therefore, in which the two segments defined by the two points are the classes K_1 and K_2 respectively, provided the extremities of the segments be assigned to the classes K_1, K_2 in any one of the possible four ways.

From Theorem 25 follows at once that the projective transform of a cut is again a cut.

Definition. Given a cut K_1, K_2 in a net of rationality on a line, and let A_1, A_2 be any two points of K_1, K_2 respectively; then a point X of the net which divides seg A_1XA_2 into two segments S_1 and S_2 such that S_1 contains only points of K_1, and S_2 only points of K_2, is called a *cut-point* of the cut.

It is evident from this definition that a cut can not have more than two cut-points.

Definition. A cut in a net of rationality on a line is said to be *closed*, if it has two cut-points in the net; it is said to be *singly open*, if it has a single cut-point in the net; and *doubly open*, if it has none.

Any closed cut K_1, K_2 with cut-points C_1, C_2 we will denote by $K(C_1, C_2)$. Such a cut clearly divides the net of rationality into two segments S_1, S_2 such that all points of S_1 are in K_1 and all points of S_2 in K_2. A singly open cut with cut-point C we will denote similarly by $K(C)$.

We now introduce continuity into the nets of rationality on a line by the following

ASSUMPTION OF CONTINUITY, C:

C. *If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that:* 1) *X_k is on l;* 2) *if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct;* 3) *if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.*

THEOREM C. *The point X_k is not a point of R.*

Proof. 1) X_k is not identical with Q, by C, 2) and C, 3). 2) Suppose X_k not identical with Q but in R, and let I be the involution* with double points Q and X_k. Then $K(Q)$ is transformed into a different cut $K'(Q)$. For if A, B are points on opposite sides of the cut $K(Q)$ and in the same seg (QAX_k), they are transformed into points of the complementary segment which are evidently on the same side of $K(Q)$. Hence, by C, 3), we should have two distinct cuts having the same X_k, which is contrary to C, 2).

The last assumption then implies the existence on some one line of more than one net of rationality, and hence by projection implies the existence of more than one net of rationality on every line. It is then in contradiction with E3′(r) (cf. end of § 3), which we mentioned as an assumption of closure for "rational" spaces.

We proceed to prove the properties expressed in Assumption C and Theorem C for every net of rationality on every line. We note first that every singly open cut in any net of rationality R' on l containing Q has associated with it a unique point. For, let $K'(Q)$ be such a cut and let Π be any projectivity on l which transforms R' into R and Q into itself. The cut $K'(Q)$ is then transformed into a $K(Q)$. By this projectivity a definite point X' of l is transformed into the point X associated with this $K(Q)$. Moreover the point X' is unique; for, if Π_1 is another projectivity transforming R' into R and $K'(Q)$ into $K_1(Q)$, then $\Pi_1 \Pi^{-1}$ is a projectivity transforming $K(Q)$ into $K_1(Q)$. The supposition that X' is not unique then leads at once to a contradiction of C, 3). *We define X' to be the point associated with $K'(Q)$.* Clearly also, with this definition, we see that if any two singly open cuts $K_1(Q)$ and $K_2(Q)$ on l are distinct, the points associated with them are distinct; and that in any projectivity on l leaving Q fixed whereby two singly open cuts are projective, the associated points are homologous.

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*An involution is defined as any projectivity on a line of period two. By "the" involution mentioned is meant the one in which the transform of any point P of the line is the harmonic conjugate of P with respect to Q and X_k. This form of statement does not assume the fundamental theorem.

Given now any singly open cut $K(Q')$ in any net of rationality on any line l', let $K(Q')$ be projected into a cut $K(Q)$ on l; the point X associated with $K(Q)$ is then the transform of a definite point X' on l' which is unique by reasoning similar to that employed in the preceding paragraph. *We define X' to be the point associated with $K(Q')$.* The properties expressed by C, 2) and C, 3) are then readily seen to hold on every line in space. The point thus associated with a singly open cut we will call the *irrational cut-point* of the cut; the other cut-point is then called *rational.* The results of the preceding paragraphs are summarized in the following:

THEOREM 27. 1) *Every singly open cut in any net of rationality on any line defines a unique irrational cut-point on the line not in the net.* 2) *If two such cuts on the same line with the same rational cut-point are distinct, the irrational cut-points are distinct.* 3) *If two singly open cuts are projective, their cut-points are homologous.*

Definition. The totality of points of a net of rationality $R(A, B, C)$, together with all the irrational cut-points defined by singly open cuts $K(C)$ in $R(A, B, C)$, is called the *chain* defined by A, B, C, and is denoted by $\mathfrak{C}(A, B, C)$. The irrational cut-points are said to be *irrational* with respect to A, B, C.

From 3) of the last theorem then follows directly:

COROLLARY. *The projective transform of any chain is a chain.*

THEOREM 28. *If P, Q, R are points of the chain defined by A, B, C, then A, B, C are points of the chain defined by P, Q, R.*

Proof. As in the proof of Theorem 16 we need only show that if P is a point of $\mathfrak{C}(A, B, C)$, then C is a point of $\mathfrak{C}(A, B, P)$ and this only when P is irrational with respect to A, B, C. Let P be defined by the singly open cut $K(C)$. This cut is transformed by the projectivity $(A, B, C, P) \overline{\wedge} (B, A, P, C)$ into a singly open cut $K(P)$ of the net $R(B, A, P)$, whose irrational cut-point must (by Theorem 27, 3)) be C.

COROLLARY 1. *A chain is determined by any distinct three of its points.*

COROLLARY 2. *A chain contains the irrational cut-point of every singly open cut in any net of rationality in the chain.*

COROLLARY 3. *Every point of* $\mathfrak{C}(A, B, C)$ *irrational with respect to* A, B, C *can be defined by a singly open cut* $K(P)$, *where* P *is any point of* $R(A, B, C)$.

We can now easily derive

THEOREM 29: THE FUNDAMENTAL THEOREM OF PROJECTIVITY FOR A CHAIN. *If* A, B, C, D *are distinct points of a chain and* A', B', C' *any three distinct points of a line, then for any projectivities giving* $(A, B, C, D) \barwedge (A', B', C', D')$ *and* $(A, B, C, D) \barwedge (A', B', C', D_1')$ *we have* $D' = D_1'$.

Proof. Let Π, Π_1 be the two projectivities mentioned in the theorem. $\Pi_1^{-1}\,\Pi$ then leaves every point of $\mathfrak{C}(A, B, C)$ fixed; for it leaves every point of $R(A, B, C)$ fixed, and hence by Theorem 27, 3) must leave every irrational cut-point of singly open cuts in $R(A, B, C)$ fixed. But $\Pi_1^{-1}\,\Pi$ is then the identical transformation as far as the points of $\mathfrak{C}(A, B, C)$ are concerned; whence $D' = D_1'$.

This theorem may also be stated as follows:

Any projective correspondence between the points of two chains is uniquely determined by three pairs of homologous points.

From this theorem follows that the points of a chain determine a commutative number-system, which by reference to Assumption C will in the next section be seen to be isomorphic with the system of ordinary real numbers.

§6. *Ordered Transformations in a Chain.*

The relation of separation between pairs of points has been defined only when the four points belong to the same net of rationality on a line. We proceed to extend the definition to any four points of the same chain.

Definition. A, B, C, D being four points of the same chain and D irrational with respect to A, B, C, the pair A, C is said to *separate* the pair B, D, if and only if A, C belong to different sides of the cut $K(B)$ of $R(A, B, C)$ defining D.

This definition is justified by Corollary 3 of Theorem 28.

This relation of separation is now defined for all the points of a chain, and is readily seen to have the fundamental properties expressed in Theorem 24. For Theorem 25 clearly holds for the more general use of the term, and this leads easily to the properties mentioned.

Definition. Given any three distinct points of a chain, the *segment* ABC of the chain (Seg ABC) consists of B and all points X of the chain such that A, C do not separate B, X; the remaining points of the chain, excluding A, C, constitute the segment of the chain complementary to Seg ABC. In the sequel the word "segment" will always mean segment of a chain, unless otherwise specified.

Clearly Seg ABC and Seg CBA contain the same points.

Any two distinct points of a chain then divide the chain into two complementary segments such that the given points separate every pair of points of the chain of which one lies in one of the segments and the other in the other segment. Conversely, whenever the points of a chain fall into two classes K_1, K_2 such that every point of the chain belongs either to K_1 or to K_2 and such that no pair of points of K_1 separates any pair of K_2, there exist two points of the chain which divide the chain into two segments S_1, S_2 such that every point of S_1 is a point of K_1 and every point of S_2 a point K_2.

We may now readily define *order* in a chain. We have seen that Seg ABC and Seg CBA contain the same points. Corresponding, however, to the two symbols $\overline{ABC}$ and $\overline{CBA}$ we distinguish two *orders* in the segment.

Definition. If two points P, Q are two points of the segment ABC of a chain, P is said to *precede* $Q\,(P<Q)$ in the order $\overline{ABC}$ if and only if $AQ \parallel PC$; Q is then said to *follow* $P\,(Q>P)$. Further, A is said to *precede* and C to *follow* every point of the segment in the order $\overline{ABC}$. The phrase "P, Q, ..., etc., are points of the directed segment $\overline{ABC}$" will in the sequel imply that P, Q, etc., are points of the segment ABC and that the statement $P<Q$ means "$P<Q$ in the order $\overline{ABC}$."

This relation of *linear order* ($<$) is at once seen to satisfy the following conditions:

THEOREM 30. 1) *If we have $P<Q$ in a given order, then $Q<P$ is impossible in that order.* 2) *If we have $P \neq Q$, then in a given order we have either $P<Q$ or $Q<P$.* 3) *If P, Q, R are points of a directed segment $\overline{ABC}$ such that we have $P<Q$ and $Q<R$, then we have $P<R$.**

From the definition of order it follows that if P precedes Q in the directed segment $\overline{ABC}$, then Q precedes P in the directed segment $\overline{CBA}$. The order in these two directed segments is therefore said to be *opposite.*

A chain is now seen to have the following fundamental property:

THEOREM 31. *If the points of a directed segment of a chain be divided into two classes H_1, H_2 such that every point of the segment belongs either to H_1 or to H_2,*

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*These three properties are sufficient to define linear order abstractly. Cf. HUNTINGTON, The Continuum as a Type of Order, *Annals of Mathematics*, Vol. VI (1905), p. 151.

and such that every point of H_1 precedes every point of H_2, then there exists one point M of the segment such that every element which precedes M is a point of H_1 and every point which follows M is a point of H_2.

Definition. A sequence of points $P_1, P_2, P_3, \ldots, P_n$ of a chain is said to be an *ordered sequence*, if they are points of a directed segment such that

$$P_1 < P_2 < P_3 \ldots < P_n.$$

Any three points of a chain are an ordered sequence, but any four points are not.

THEOREM 32. *If A, B, C, D are an ordered sequence, so also are B, C, D, A.*

Proof. By definition we have $AC \| BD$ in the directed segment $\overline{ACD}$; whence in the directed segment $\overline{BDA}$, we have $BD \| CA$.

COROLLARY. *If $P_1, P_2, P_3, \ldots, P_{n-1}, P_n$ form an ordered sequence, so likewise do $P_i, P_{i+1}, P_{i+2}, \ldots, P_n, P_1, P_2, P_3, \ldots, P_{i-1}$* and *$P_i, P_{i-1}, \ldots, P_2, P_1, P_n, P_{n-1}, \ldots, P_{i+1}$.*

Hence, given any ordered sequence of points of a chain and starting with any one of the points, it is possible to write them so as to form an ordered sequence in two and only two ways. This is expressed by saying that we can take the points in two different *directions*, which are opposite.*

Definition. A transformation which transforms every ordered sequence into an ordered sequence is called an *ordered transformation.* In all that follows, the word transformation denotes a correspondence which is single-valued (one-to-one) and whose inverse is also single-valued.

From Theorem 25 we have at once:

THEOREM 33. *Every projectivity on a line is an ordered transformation.*

Definition. In the number-system determined by the scale $(0, 1, \infty)$ on a chain a number a is said to be *less than* a number b, if $a < b$ in the order $\overline{01\infty}$.

THEOREM 34. *The number-system determined by the scale* $(0, 1, \infty)$ *in a chain is isomorphic with the system of real numbers.*

Proof. This theorem may be conveniently established by referring to a set of postulates descriptive of the real number-system. We shall use the set given by HUNTINGTON in Vol. VI (1905), p. 39, of the *Transactions of the American*

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*This establishes the so-called "cyclical order" in a chain. Cf. ENRIQUES' assumption, *Vorlesungen über projektive Geometrie*, Leipzig (1903), p. 23.

Mathematical Society. That HUNTINGTON'S *I, II, A*1–*A*6, *M*1, *M*2, *AM*1 are satisfied is equivalent to the fact that we have to do with a commutative number-system, which is a consequence of Theorem 29. In consequence of the definition above and Theorem 30 the elements of this number-system satisfy the magnitude relations "greater and less than" and the postulate of continuity. This verifies HUNTINGTON'S *R*1–*R*6. The projectivities $x' = x + a$ and $x' = ax$ transform Seg $(-a\,0\,\infty)$ into Seg $(0\,a\,\infty)$ and Seg $(0\,1\,\infty)$ into Seg $(0\,a\,\infty)$ respectively. This, in connection with Theorem 33, shows that if $a > 0$ and $b > 0$ then $a + b > 0$ and $ab > 0$. In like manner if $a < 0$ and $b < 0$ then $a + b < 0$. This verifies HUNTINGTON'S *RA*1, *RA*2, *RAM*1, and completes the list of assumptions which he uses to characterize the system of real numbers categorically.

We consider now a projective transformation of a chain into itself. Such a transformation is ordered, but the directions of the transformed sequences may or may not be the same as those of the original sequences. If the direction in a chain is preserved by a transformation, the latter will be called *directly ordered*, or simply *direct;* otherwise, if ordered, it is *oppositely ordered*, or *opposite.*

The analytical condition that a projective transformation of a chain into itself be direct or opposite is now readily obtained. Let the chain be $\mathfrak{C}(01\infty)$. We have already seen that for any class of points forming a commutative number-system any projectivity is given by

$$x' = \frac{ax + b}{cx + d}, \qquad D = ad - bc \neq 0. \tag{1}$$

If $\mathfrak{C}(01\infty)$ is transformed into itself it is clear that a, b, c, d are all real numbers. The projectivity $x' = x + b$ is direct, $x' = \frac{1}{x}$ is opposite, while $x' = ax$ is direct or opposite according as a is positive or negative. The desired condition given in the following theorem is then obtained at once by recalling the theorem that the determinant of the product of two projectivities is equal to the product of their determinants.

THEOREM 35. *A projective transformation* (1) *transforming* $\mathfrak{C}(01\infty)$ *into itself is direct or opposite according as D is positive or negative.*

Definition. A point which is made to correspond to itself by a transformation is called a *double point* of the transformation. A projectivity which transforms a chain into itself is said to be *hyperbolic, parabolic,* or *elliptic in the chain* according as it has two, one, or no double points in the chain.

The double points of a projectivity (1) transforming $\mathfrak{C}(01\infty)$ into itself, if they exist, are given by the roots of the equation $cx^2 + (d - a)x - b = 0$, where a, b, c, d are real. This equation has roots in the chain if and only if the discriminant

$$\begin{aligned}\Delta &= (d - a)^2 + 4bc \\ &= (d + a)^2 - 4D\end{aligned}$$

is positive or zero.

From this follows at once:

THEOREM 36. *In a chain,* 1) *every opposite projectivity is hyperbolic;* 2) *every parabolic or elliptic projectivity is direct.*

The proof of this theorem demands at some point a continuity argument We have chosen to borrow the desired result from the theory of functions of a real variable. It can, however, be proved directly from our assumptions without difficulty. We refer for this proof to ENRIQUES, *Vorlesungen über Projektive. Geometrie,* Leipzig (1903), pp. 72, 100.

Also from the definitions preceding we have at once:

THEOREM 37. *A hyperbolic projectivity in a chain is opposite or direct according as pairs of homologous points do or do not separate the double points.*

From the consideration of the fundamental cross-ratio it follows easily that if an involution (*i.e.,* a projectivity of period two) which transforms a chain into itself has a double point in the chain, it has another, and that the double points separate harmonically every pair of conjugate points. From the last two theorems and Theorem 26, corollary, then follows:

THEOREM 38. *An involution in a chain is direct and elliptic in the chain or opposite and hyperbolic, according as two pairs of conjugate points do or do not separate each other.*

Since an involution in a chain is determined by two pairs of conjugate points, the existence of both kinds of involutions follows.

§ 7. *The Ordinary Real and Complex Projective Spaces.*

We can now conveniently add the further assumptions necessary to characterize completely 1) the ordinary real projective space, or 2) the ordinary complex projective space, of three dimensions. Analytically this is equivalent to the identification of our number-system with 1) the system of ordinary real numbers, or 2) the system of ordinary complex numbers.

1). To characterize the ordinary real projective space we add simply the following assumption (of closure):

ASSUMPTION R. *There is not more than one chain on a line.*

A fundamental consequence of this assumption is the existence of projectivities on a line without double points. In fact any involution on the line determined by two pairs of conjugate points which separate each other is of this kind (Theorem 38).

2). On the other hand, to characterize completely the ordinary complex projective space we need only *replace* ASSUMPTION R by the following, ASSUMPTIONS I.

ASSUMPTION I1. *If there is a harmonic form, there is one $(ABA'B')$ such that one involution I having AA' and BB' as conjugate pairs has a double point on the line AB.*

By Theorem 26, corollary, and Theorem 38, the involution I has no double points in the chain $\mathfrak{C}(ABA')$; this assumption then implies the existence of more than one chain on the line AB. Assumptions R and I1 are then mutually contradictory (in connection with the assumptions already made).

ASSUMPTION I2.* *Through a point P of a chain $\mathfrak{C}$ on a line l and any point J of l not on $\mathfrak{C}$ there is not more than one chain that has no other point in common with $\mathfrak{C}$ than P.*

We are now in a position to prove that our number-system is indeed isomorphic with the ordinary system of complex numbers. We will show first that every point of the line l is given by the expression $A + JB$, where A, B are points of $\mathfrak{C}$ and J is a fixed point not on $\mathfrak{C}$.

Let the point P of $\mathfrak{C}$ be labelled ∞ and let any pair of points of $\mathfrak{C}$ distinct from P be labelled 0 and 1 respectively. The points of $\mathfrak{C}$ are then isomorphic with the system of real numbers and ∞ (Theorem 34). *Without assuming the commutativity of multiplication* it is readily seen that

$$x' = x + a, \quad x' = ax, \quad x' = xa, \quad x' = x^{-1}$$

each define a projectivity when a is constant. This follows easily from the Figs. 2, 3 and 6 (cf. Theorem 12). The totality of points $A + J$, where A stands for any point of $\mathfrak{C}$, therefore constitutes a chain $\mathfrak{C}_1$, by Theorem 27, corollary. This chain has no point in common with $\mathfrak{C}$ besides P, because if $A + J = B$, where $B(\neq \infty)$ is a point of $\mathfrak{C}$, we should have $B - A = J$, which would make J a point of $\mathfrak{C}$.

---

* I2 is an assumption of closure.

Now let X be any point of l not in $\mathfrak{C}$ or $\mathfrak{C}_1$. The chain $\mathfrak{C}(XJP)$ has by 12 a point x_1, $\neq P$, in common with $\mathfrak{C}$. The projectivity $x' = x + J(1 - x_1^{-1}x)$, $(x_1 \neq 0)$, transforms $\mathfrak{C}$ into $\mathfrak{C}(XJP)$, so that every point of the latter and hence X is of the form $A + JB$, where A, B are points of $\mathfrak{C}$. If $x_1 = 0$, the projectivity $x' = Jx$ shows likewise that X is of the desired form $(A = 0)$. The points of $\mathfrak{C}_1$ also have this form. The desired result is then established.

We shall now prove the fundamental theorem of projectivity for all the points of our complex line by showing that the number-system determined on the line is commutative; that the latter is isomorphic with the system of ordinary complex numbers will then follow at once.

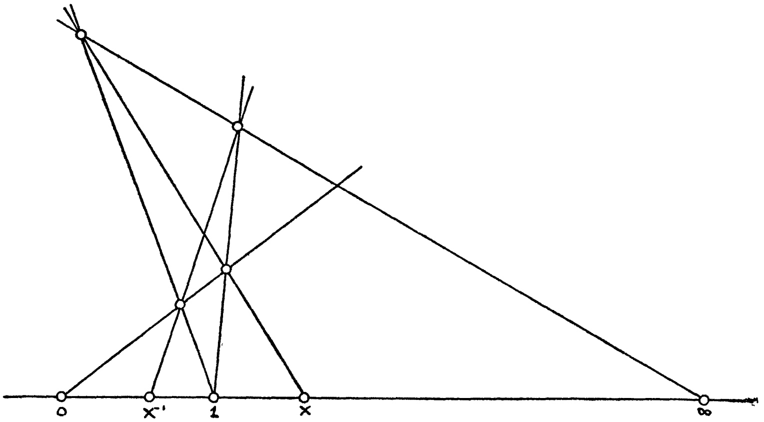

FIG. 6.

Let the points A, B, A' of 11 be labelled $0, 1, \infty$, so that the chain $\mathfrak{C}(ABA')$ is made isomorphic with the system of ordinary real numbers (and ∞), and let the double point of I in 11 be denoted by i. By the result just established all the points of the line are of the form $x + iy$, where x, y are real, since i is not on the chain $\mathfrak{C}(ABA')$. Moreover, two points $a + ib$ and $c + id$ are identical, if and only if $a = c$ and $b = d$, if a, b, c, d are real; for the equality $a + ib = c + id$ implies the relation $i = (c - a)(b - d)^{-1}$, if $b - d \neq 0$. Now, each of the projectivities $x' = ix$ and $x' = xi$, evidently transforms the chain $\mathfrak{C}(01\infty)$ into the chain $\mathfrak{C}(0i\infty)$; this gives $xi = ix_1$, where x, x_1 are real. Also each of the projectivities $x' = (1 - i)x$ and $x' = x(1 - i)$ transforms $\mathfrak{C}(01\infty)$ into $\mathfrak{C}(0, 1 - i, \infty)$, whence at once

$$x(1 - i) = (1 - i)x_2,$$

$x_1 x_2$ being real, or by the distributive law (Theorem 13),

$$x - xi = x_2 - ix_2,$$

or by the above,

$$x - ix_1 = x_2 - ix_2,$$

or finally,

$$x = x_2 = x_1.$$

This gives $xi = ix$, for any real x, and hence follows readily the *commutativity of multiplication for any two of the numbers* $x + iy$. This in connection with Theorem 14 proves the following:

THEOREM 39. *The fundamental theorem of projectivity holds for all the points of a complex line.*

For, if it is valid on one line it is valid on every line by projection.

In view of the last theorem I is the only involution having AA' and BB' as conjugate pairs and is given by $x' = -\frac{1}{x}$; this gives at once $i^2 = -1$, and completes the proof of:

THEOREM 40. *The number-system on a line in the complex space is isomorphic with the system of all complex numbers and* ∞.

It is interesting to note here the well-known fact that whereas the property of transforming any quadrangular set into such a set is necessary and sufficient to characterize projective transformations on a line in the real geometry, it is not sufficient in the complex.

Suppose we have a transformation f which leaves the points 0, 1, ∞ fixed and transforms quadrangular sets into quadrangular sets. It is then necessarily an *ordered transformation* subject to the following functional conditions:

$$f(x+y)=f(x)+f(y),\ f(xy)=f(x)f(y),\ f(0)=0,\ f(1)=1,\ f(\infty)=\infty.$$

From the equation $f(x+1) = f(x) + 1$ then follows at once that $f(a) = a$, where a is any positive integer; from $f(x) + f(-x) = 0$ follows the same relation when a is any negative integer; from $f(x)\,f(1/x) = 1$ then follows readily $f(x/y) = f(x)/f(y)$, whence follows at once the relation $f(a) = a$, where a is any rational fraction or zero. From the last relation and the fact that f is ordered then follows at once the fact that *f leaves every real number fixed.* But this is sufficient to identify any transformation which transforms quadrangular sets into such sets with a projectivity on the real line. For the complex line we have at once $f(x + iy) = x + f(i)y$. Let $f(i) = a + ib$, where a and b are real, then $f(i)\,f(i) = -1$ gives $a^2 - b^2 + 1 + 2abi = 0$, whence $a = 0$, or

$b = 0$; the latter leads to the impossible relation $a^2 + 1 = 0$; the former gives $f(i) = \pm i$. By Theorem 40, $f(i) = i$ alone gives a projectivity; the relation $f(i) = -i$ leads to the so-called *anti-projectivities* of SEGRE.*

§ 8. *Categorical Systems. Quadratic Irrationalities.*

It is now very easy to see that our sets of assumptions for real and for complex projective geometry are categorical.† Confining our attention to the real case, it is clear that in any space satisfying assumptions A, E, H, C, R it is possible to establish a system of homogeneous coordinates such that every point is denoted by the ratios $x_1 : x_2 : x_3 : x_4$, where the x_i are real numbers. Therefore, given any two such spaces, a one-to-one reciprocal correspondence is set up between them in such a way as to preserve all geometrical relations, provided each point in one space corresponds to a point with the same coordinates in the other space. Since it is possible to choose the tetrahedron of reference and the point (1, 1, 1, 1) in ∞^{15} ways (corresponding to the collineations of the general projective group), we have the following:

THEOREM 41. *Any two spaces which satisfy assumptions* A, E, H, C, R *are simply isomorphic in* ∞^{15} *ways.*

In like manner is proved:

THEOREM 42. *Any two spaces which satisfy assumptions* A, E, H, C, I *are simply isomorphic in* ∞^{15} *ways.*

The following considerations will help to make clear the bearing of Assumptions C, R and I. The points and lines of a two-dimensional net of rationality form in their relations among themselves a projective plane (Theorem 21) and may be discussed either by synthetic methods or by an analytic geometry in which the coordinates are rational numbers. Corresponding lines of two projective non-perspective pencils of lines in the net intersect in a set of points in the net which lie on a conic section. This conic is said to *belong to the net.* Denote such a conic by C. Let us now recall the definitions of addition and multiplication (p. 353) which require that x and y, 0 and $x + y$ shall be pairs of an involution of which ∞ is a double point, and that x and y, 1 and xy, 0 and ∞ shall be pairs of another involution. Projecting these involutions upon the conic C we have by a familiar theorem that the lines joining corresponding points of the involutions must meet in a point, through which pass the tangents at the double points, provided double points exist (cf. Figs. 7, 8). Therefore to construct the square roots of a number z, it is necessary to construct the tangents to C from the point of intersection of the lines 0∞ and $1z$. If z is -1, then 1 and -1 harmonically divide 0 and ∞, *i. e.*, the line 0∞ passes through the point of intersection of the tangents at 1 and -1. The existence of $\sqrt{-1}$ depends, therefore, upon the possibility of drawing a tangent to a conic section from the point P of intersection of two chords of the conic, each of which passes through the polar point of the other. Assumption I1 states that this is possible. Assumption R states that it is not possible. In the geometry in which R holds P is an inside point of the conic.

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* SEGRE, Un Nuovo Campo di Ricerche Geometriche, *Torino Atti*, Vol. XXV (1890), pp. 276, 430; Vol. XXVI (1891), pp. 35, 592.

† For a discussion of this mathematico-logical term see VEBLEN, HUNTINGTON'S Types of Serial Order, *Bull. American Math. Soc.*, Vol. XII (1906), p. 303.

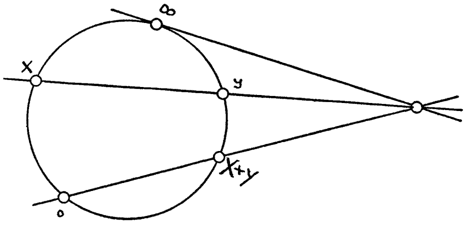

FIG. 7.

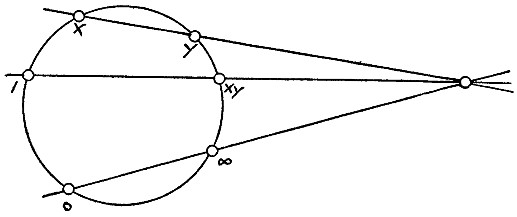

FIG. 8.

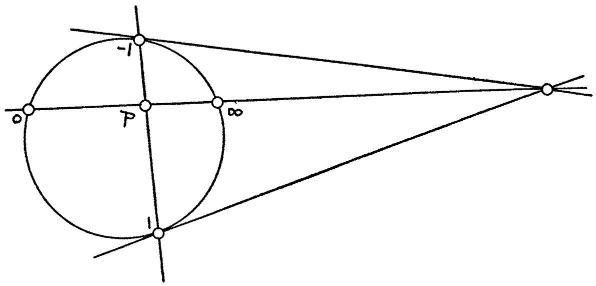

FIG. 9.

For a conic associated with a net of rationality, as C is above, the interior and exterior may be defined as follows: The order of the rational points of the conic having been determined by projection from the order of the rational points on any line, draw two lines through any point P not on the conic, the first meeting the conic in A_1, A_2, and the second meeting the conic in B_1, B_2. If A_1, A_2 separate B_1, B_2 the point P is an interior point; if not, an exterior point.

Many of the purposes of elementary projective geometry are served by operations which do not introduce into the coordinates of the points considered irrationalities of more than a certain degree. For such purposes it is not necessary to assume as much as Assumption C. The presence of the rational points is assured by Assumptions A, E and H. To adjoin to this field the operation $\sqrt{x}$, where x is positive, we may assume instead of C:

If P is any point of a two-dimensional net of rationality R exterior to a conic C belonging to the net, then there is at least one tangent to C which passes through P

Equivalent statements to this are:

An involution in which points of a given net of rationality are paired with points of the same net, and in which two conjugate pairs do not separate each other, has at least one double point. (The latter statement is readily seen to be equivalent to the former by letting the involution lie on a conic.)

A line joining two points of R, one interior to C and one exterior, meets C in at least one point.

§ 9. *List of Assumptions and their Mutual Independence.*

The following is a list of our assumptions for ordinary real projective space. The page references are to the definitions of terms occurring in the assumptions.

A1. *If A and B are distinct points, there is at least one line containing both A and B.*

A2. *If A and B are distinct points, there is not more than one line containing both A and B.*

A3. *If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.*

E0. *There are at least three points on every line.*

E1. *There exists at least one line.*

E2. *It is not true that every point lies on every line.*

E3. *It is not true that every point lies on every plane.* (P. 349.)

E3′. *If S is a three-space, every point lies in S.* (P. 349.)

H. *If there is a harmonic sequence, there is one such that every point of it is distinct from all the points of the sequence that precede it.* (P. 359.)

C. *If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that:* 1) *X_k is on l;* 2) *if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct;* 3) *if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.* (Pp. 356, 363.)

R. *There is not more than one chain on a line.* (P. 365.)

For the ordinary complex projective space, Assumption R is replaced by the following two:

I1. *If there is a harmonic form, there is one $(ABA'B')$ such that one involution having AA' and BB' as conjugate pairs has a double point on the line AB.* (Pp. 351, 364, footnote.)

I2. *Through a point P of a chain $\mathfrak{C}$ on a line l and any point J of l not in $\mathfrak{C}$ there is not more than one chain that has no other point in common with $\mathfrak{C}$ than P.* (P. 365.)

We are now to prove that the assumptions above given are mutually independent, *i. e.*, such that no one of them is a formal logical consequence of the remaining ones. The method of doing this is fully explained in connection with Assumption A1, and is only sketched in the other cases.

ASSUMPTION A1. Consider the four letters A, B, C, P. Call them pseudo-points and call the set of three A, B, C a pseudo-line. The whole set A, B, C, P may be called a pseudo-space. Now, if the words "point" and "line" in the assumptions are taken to refer to these pseudo-points and line, it is evident that A1 is a false proposition, because there is no line containing both A and P. On the other hand A2 is a true proposition because there is only one line in all. A3 is true, though trivial. E0, E1, E2 are clearly true; E3 is true because no plane exists (cf. definition, p. 349). The hypotheses of Assumptions E3′, H, C, I_1 and I_2 are not satisfied by our pseudo-space. To introduce a technical phrase due to HUNTINGTON for the condition here met, Assumptions E3′, H, I, and C are "vacuously satisfied," or, as we may say more briefly, are "vacant." Clearly Assumption R is true.

Now any proposition which is a logical consequence of Assumptions A2, A3, E, H, C and R (or I) either must be true of our pseudo-space or may be vacant because involving in its deduction one or more of the vacant assumptions. The proposition A1 is neither true nor vacant of our pseudo-space, but false. Therefore A1 is not a logical consequence of the other assumptions.

ASSUMPTION A2. Let the pseudo-space consist of the points of an ordinary plane, and let all the usual lines be pseudo-lines, but in addition to these let all the points of the plane constitute a pseudo-line. In this pseudo-space every three points are collinear; hence there exists no plane. It is then readily seen that Assumptions A1, E0, E1, E2 and R are true, while A3, E3, E3$'$, H, C and I are vacant. Clearly also A2 is false for this pseudo-space. This proves A2 independent of all the other assumptions.

ASSUMPTION A3. Let the pseudo-points consist of the nine digits 1, 2, ... , 9; and let each row, each column, and each term in the determinant expansion of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

constitute a pseudo-line. In this pseudo-space A3 is false, as can readily be verified. A1, A2, E0, E1, E2, E3, E3$'$ and R are true; H, C, I1 and I2 are vacant.

ASSUMPTION E0. Let the pseudo-space consist of four pseudo-points, where the pseudo-lines are the pairs of pseudo-points. It follows that the planes are triples of pseudo-points. E0 is false, H, C and I1, I2 are vacant, while all the other assumptions are true.

ASSUMPTION E1. Let the pseudo-space consist of one pseudo-point and no pseudo-lines. All the assumptions are vacant except E1, which is false.

ASSUMPTION E2. Let the pseudo-space consist of three pseudo-points A, B, C and one pseudo-line ABC. Here E2 is false. A1, A2, E0, E1 and R are true; A3, E3, E3$'$, H, C, I1 and I2 are vacant.

ASSUMPTION E3. Let the pseudo-space consist of all the points of a single real (complex) projective plane, and let the pseudo-lines consist of the lines of this plane. All the assumptions for real (complex) projective geometry are true except E3, which is false, and E3$'$, H, C, I, which are vacant.

ASSUMPTION E3$'$. Let the pseudo-space be an ordinary real (complex) projective space of four dimensions. Its points may, for example, be described analytically as consisting of all sets of five homogeneous real (complex) coordinates $(x_1, x_2, x_3, x_4, x_5)$, except $(0, 0, 0, 0, 0)$, the lines being the sets of all points which satisfy three linear homogeneous equations. For such a space all the assumptions for real (complex) geometry are true, except E3$'$, which is false.

ASSUMPTION H. Let the pseudo-space consist of all sets of four homogeneous coordinates which are ordinary integers reduced modulo 2. In this pseudo-space H is false, C, I1 and I2 are vacant, while all the other assumptions are true.*

ASSUMPTION C. Let the pseudo-space consist of all sets of four homogeneous coordinates, except (0, 0, 0, 0), which consist of rational numbers only. Since all the assumptions for real geometry are true of this space except C, this proves the desired independence in case of the real geometry. For the complex geometry let the coordinates consist of all numbers of the form $A + Bi$, where A, B are rational. That parts 2) and 3) of C are independent of 1) and the other assumptions may be seen as follows:

C, 2). Let $[x]$ be a set of irrational numbers, such that every irrational number is of the form $ax + b$, where a, b are rational, and such that none of the numbers x is rationally related to any other x'; *i. e.*, that there is no relation of the form $x' = ax + b$, where a and b are rational.† Now, let the pseudo-space consist of a three-dimensional projective space of points, whose coordinates are rational complex numbers. Using non-homogeneous coordinates let the line l be the line $y = 0$, $z = 0$ and let the point Q be ∞. Then with the cut in the ordinary rational numbers which determines the number $ax + b$ in the ordinary geometry associate the number $a + ib$. The same number $a + ib$ is then associated with an infinitude of distinct cuts, contrary to C, 2). All the other assumptions, including C, 1) and C, 3), are satisfied except I2, which is vacant.

C, 3). Let the pseudo-space consist of the points of ordinary real or complex projective space, and let $K_1(Q)$ and $K_2(Q)$ be any two singly open cuts on l, and X_{k_1} and X_{k_2} the cut-points determined by them in the ordinary geometry. In the pseudo-space associate X_{k_1} with $K_2(Q)$ and X_{k_2} with $K_1(Q)$, and let all other irrational points be associated with their proper cuts in the ordinary way. C, 3) is then false, while the other two parts of C and all the other assumptions for the real or complex projective geometry remain true.

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* For a detailed discussion of such finite spaces, cf. VEBLEN and BUSSEY, *Trans. Am. Math. Soc.*, Vol. VII (1906), pp. 241–259.

† The assumption of the existence of a set $[x]$ is closely related to ZERMELO'S assumption of the existence of an "ausgezeichnetes Element" in any class, though our assumption is weaker. It may be stated as follows: Let $R(x)$ denote the class of all numbers of the form $ax+b$, where a and b are any rational numbers, and x is a given irrational number. Any two distinct classes $R(x)$ are then mutually exclusive. Consider the class of al classes $R(x)$. Our assumption above then states that there exists a class $[x]$ of numbers which contains one and only one number from each of the classes $R(x)$, and no others. The class of classes $R(x)$ has the same cardinal (Mächtigkeit) as the continuum, whereas ZERMELO'S assumption has reference to the class of all sub-classes of the continuum, whose cardinal is greater than that of the continuum.

ASSUMPTION R. Let the pseudo-space consist of the three-dimensional projective space in which the coordinates are ordinary complex numbers.

ASSUMPTION I1. Let the pseudo-space consist of the three-dimensional projective space, in which the coordinates are ordinary real numbers. All the other assumptions for the complex geometry are true, except I1, which is false, and I2, which is vacant.

ASSUMPTION I2. Let the pseudo-space consist of all sets of four homogeneous coordinates, excluding $(0, 0, 0, 0)$, which are marks of a field, F, consisting of the ordinary complex numbers together with an additional unit, t, and all algebraic functions of these.

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